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Asymptotics of zeros of incomplete gamma functions

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Dedicated to Luigi Gatteschi, in admiration for his excellent contributions to the study of zeros of orthogonal polynomials

We consider the complex zeros with respect to z of the incomplete gamma functions $\gamma(a,z)$ and $\Gamma(a,z)$, with a real and positive. In particular we are interested in the case that a is large. The zeros are obtained from approximations that are computed by using uniform asymptotic expansions of the incomplete gamma functions. The complex zeros of the complementary error function are used as a first approximation. Applications are discussed for the zeros of the partial sums $s_n(z) = \sum_{j=0}^{n} z^j/j!$ of $\exp(z)$.

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1. Introduction

The incomplete gamma functions are defined by

$$\gamma(a,z) = \int_0^z t^{a-1}e^{-t}\,dt, \quad \Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}\,dt, \tag{1.1}$$

and $P(a,z) = \gamma(a,z)/\Gamma(a)$, $Q(a,z) = \Gamma(a,z)/\Gamma(a)$, with

$$P(a,z) + Q(a,z) = 1. \tag{1.2}$$

We take $a > 0$ and z complex, $|\arg z| < \pi$. The function $\gamma^*(a,z) = z^{-a}P(a,z)$ is an entire function of both parameters a and z. When a is a non-negative integer the incomplete gamma functions are very simple:

$$\begin{aligned} \gamma(n+1,z) &= n![1 - e^{-z}s_n(z)], \\ \Gamma(n+1,z) &= n!\,e^{-z}s_n(z), \end{aligned} \qquad n = 0,1,2,\ldots, \tag{1.3}$$

in which $s_n(z)$ is the first part of the Taylor series of the exponential function:

$$s_n(z) = \sum_{j=0}^{n} \frac{z^j}{j!}, \quad n = 0,1,2,\ldots. \tag{1.4}$$

We consider the zeros with respect to z of the incomplete gamma functions, with a fixed and positive. The method for obtaining the zeros is based on asymptotic expansions of the incomplete gamma functions, that hold for large values of a and are uniformly valid with respect to z. We derive asymptotic expansions of the zeros. The expansions are of the same kind as the expansion of the positive solution $z(a, q)$ of the inversion problem $Q(a, z) = q$ with $q \in (0, 1)$; see Temme [4]. This inversion problem is of importance in probability theory and mathematical statistics.

In Kölbig [2], the zeros of the incomplete gamma function $\gamma(a, z)$ are considered from a different point of view. That paper gives in particular information on the zeros with respect to a, with z real and positive. It gives curves of the zeros of $\gamma(xw, x)$ in the w-plane; the trajectories are parameterised by the positive parameter x. Kölbig's paper contains many references to earlier papers.

The asymptotic distribution of zeros of the partial sums $s_n(z)$ of the exponential function (see (1.3), (1.4)) has received much attention in the literature. For a recent overview we refer to Varga [6].

2. Uniform asymptotic expansions of P and Q

We obtain an asymptotic expansion of the zeros of the incomplete gamma functions using uniform asymptotic expansion of these functions as given in Temme [3]. First we summarize these results.

The incomplete gamma functions have the following representations

$$\begin{aligned} P(a,z) &= \tfrac{1}{2}\,\mathrm{erfc}(-\eta\sqrt{a/2}) - R_a(\eta), \\ Q(a,z) &= \tfrac{1}{2}\,\mathrm{erfc}(\eta\sqrt{a/2}) + R_a(\eta); \end{aligned} \tag{2.1}$$

erfc is the error function defined by

$$\mathrm{erfc}\, z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt. \tag{2.2}$$

The real parameter η in (2.1) is defined by

$$\tfrac{1}{2}\eta^2 = \lambda - 1 - \ln\lambda, \quad \lambda = z/a, \quad \mathrm{sign}(\eta) = \mathrm{sign}(\lambda - 1). \tag{2.3}$$

The condition on the sign of η holds for positive values of z. For complex values we define the branch of the square root by analytic continuation. Alternatively,

$$\eta = (\lambda - 1)\sqrt{2\frac{\lambda - 1 - \ln\lambda}{(\lambda - 1)^2}}, \quad \lambda = z/a, \tag{2.4}$$

where the square root takes the value 1 when $\lambda = 1$.

For the function $R_a(\eta)$ we derived an asymptotic expansion. Writing

$$R_a(\eta) = \frac{e^{-a\eta^2/2}}{\sqrt{2\pi a}} S_a(\eta), \tag{2.5}$$

we have

$$S_a(\eta) \sim \sum_{n=0}^{\infty} \frac{C_n(\eta)}{a^n}, \quad \text{as } a \to \infty. \tag{2.6}$$

This expansion holds in an unbounded domain of the parameter $\eta \in \mathbb{R}$; in particular, the expansion holds in a large neighbourhood of the point $\eta = 0$, which corresponds with $z = a$, a transition point in the asymptotic behaviour of the incomplete gamma functions for large values of the parameter a. When a is positive, the expansion (2.6) holds uniformly with respect to z in the complex plane.

The first two coefficients in (2.6) are

$$\begin{aligned} C_0(\eta) &= \frac{1}{\lambda - 1} - \frac{1}{\eta}, \\ C_1(\eta) &= \frac{1}{\eta^3} - \frac{1}{(\lambda-1)^3} - \frac{1}{(\lambda-1)^2} - \frac{1}{12(\lambda-1)}. \end{aligned} \tag{2.7}$$

These two (and all higher coefficients) have a removable singularity at $\eta = 0$ $(\lambda = 1, z = a)$. All $C_n(\eta)$ are analytic at $\eta = 0$. The higher coefficients can be obtained from the recursion

$$\eta C_n(\eta) = \frac{d}{d\eta} C_{n-1}(\eta) + \frac{\eta}{\lambda - 1}\gamma_n, \quad n \geqslant 1, \tag{2.8}$$

where the numbers γ_n appear in the well-known asymptotic expansion of the Euler gamma function. That is,

$$\Gamma^*(a) \sim \sum_{n=0}^{\infty} (-1)^n \gamma_n a^{-n}, \quad \frac{1}{\Gamma^*(a)} \sim \sum_{n=0}^{\infty} \gamma_n a^{-n}, \quad a \to \infty, \tag{2.9}$$

where

$$\Gamma^*(a) = \sqrt{\frac{a}{2\pi}} e^a a^{-a} \Gamma(a), \quad a > 0. \tag{2.10}$$

The first few γ_n are

$$\gamma_0 = 1, \quad \gamma_1 = -\tfrac{1}{12}, \quad \gamma_2 = \tfrac{1}{288}, \quad \gamma_3 = \tfrac{139}{51840}.$$

3. Asymptotic expansions of the zeros of the incomplete gamma functions

We concentrate on the function $Q(a, z)$. We first compute the zeros in terms of the parameter η, by using the representation in (2.1), with large values of a. Afterwards we compute λ and z from (2.3).

We first repeat the steps in the inversion problem

$$\tfrac{1}{2}\operatorname{erfc}(\eta(\sqrt{a/2}) + R_a(\eta) = q, \tag{3.1}$$

which has been considered in Temme [4] when $q \in [0, 1]$. The procedure started with the solution η_0 of the equation

$$\tfrac{1}{2}\operatorname{erfc}\left(\eta_0\sqrt{a/2}\right) = q, \tag{3.2}$$

and the solution η defined by (3.1) is written as $\eta(q, a) = \eta_0(q, a) + \epsilon(\eta_0, a)$. The quantity $\epsilon(\eta_0, a)$ is written in the form of the expansion

$$\epsilon(\eta_0, a) \sim \frac{\epsilon_1}{a} + \frac{\epsilon_2}{a^2} + \frac{\epsilon_3}{a^3} + \dots, \tag{3.3}$$

as $a \to \infty$. The coefficients ϵ_i can be written explicitly as functions of η_0.

In [4] a differential equation for ϵ was derived from which the expansion (3.3) could be obtained by perturbation methods. The same differential equation can be used in the present problem to determine the zeros. Observe that (3.1) yields the relation

$$\frac{dq}{d\eta} = \frac{d}{d\eta} Q(a, z) = \frac{d}{dz} Q(a, z) \frac{dz}{d\eta}.$$

Using (1.1) and (2.3), one obtains after straightforward calculations

$$\frac{dq}{d\eta} = -\frac{1}{\Gamma^*(a)}\sqrt{\frac{a}{2\pi}} f(\eta)\, e^{-a\eta^2/2}, \tag{3.4}$$

where $\Gamma^*(a)$ is defined in (2.10), and

$$f(\eta) = \frac{\eta}{\lambda - 1}, \tag{3.5}$$

the relation between η and λ being given in (2.3). For small values of η we can expand

$$f(\eta) = 1 - \tfrac{1}{3}\eta + \tfrac{1}{12}\eta^2 + \dots . \tag{3.6}$$

From (3.2) it follows that

$$\frac{dq}{d\eta_0} = -\sqrt{\frac{a}{2\pi}}\, e^{-a\eta_0^2/2}. \tag{3.7}$$

Upon dividing the two differential equations in (3.4), (3.7), the quantity q is eliminated. That is,

$$\frac{d\eta}{d\eta_0} = \frac{\Gamma^*(a)}{f(\eta)}\, e^{a(\eta^2 - \eta_0^2)/2}. \tag{3.8}$$

Replacing η with $\eta_0 + \epsilon$, one has

$$f(\eta_0 + \epsilon)\left[1 + \frac{d\epsilon}{d\eta_0}\right] = \Gamma^*(a)\, e^{a\epsilon(\eta_0 + \epsilon/2)},$$

a relation between ϵ and η_0, with a as (large) parameter.

As remarked earlier, we use this differential equation also in the present case. The value $q = 0$ gives in our earlier paper the trivial solution $\eta_0 = \infty$, i.e., $z = +\infty$; this solution is not considered in the present case. Obviously, the above approach does not require the condition $q \in [0, 1]$. In fact, q may assume any complex value; when we take $q = 0$ the quantity η_0 is associated with the finite zeros of the complementary error function in (2.1). More information on these zeros will be given in the following section.

The quantities ϵ_i introduced in (3.3) can be computed as in [4]. Let η_0 be a finite complex number satisfying (3.2), with a a given positive number. Then η_0 will be a first order approximation (for large values of a) of η, that solves the equation

$$\tfrac{1}{2}\operatorname{erfc}(\eta\sqrt{a/2}) + R_a(\eta) = q. \tag{3.9}$$

This equation is equivalent with $Q(a, \lambda a) = 0$, the relation between λ and η being given in (2.4). Higher order approximations to η are obtained by writing

$$\eta \sim \eta_0 + \frac{\epsilon_1}{a} + \frac{\epsilon_2}{a^2} + \frac{\epsilon_3}{a^3} + \cdots. \tag{3.10}$$

The first few ϵ_i are given by

$$\begin{aligned}
\epsilon_1 &= \frac{1}{\eta}\ln f(\eta),\\
12\eta^3\epsilon_2 &= +12 - 12f^2 - 12f\eta - 12f^2\eta\epsilon_1 - 12f\eta^2\epsilon_1 - \eta^2 - 6\eta^2\epsilon_1^2,\\
12\eta^5\epsilon_3 &= -30 + 12f^2\eta\epsilon_1 + 12f\eta^2\epsilon_1 + 24f^2\eta^3\epsilon_1 + 6\epsilon_1^3\eta^3 - 12f^2 + 60f^3\eta^2\epsilon_1\\
&\quad + 31f^2\eta^2 + 72f^3\eta + 42f^4 + 18f^3\eta^3\epsilon_1^2 + 6f^2\eta^4\epsilon_1^2 + 36f^4\eta\epsilon_1\\
&\quad + 12\epsilon_1^2\eta^3 f + 12\epsilon_1^2\eta^2 f^2 - 12\eta\epsilon_1 + \eta^3\epsilon_1 + f\eta^3 - 12f\eta + 12\epsilon_1^2\eta^2 f^4.
\end{aligned}$$

Here, the quantity η equals η_0 and f is the function $f(\eta)$, which is defined in (3.5). For small values of η_0 it is convenient to have series expansions of the coefficients ϵ_i (note that η_0 may be quite small, when a is large). From [4] we have

$$\begin{aligned}
\epsilon_1 &= -\tfrac{1}{3} + \tfrac{1}{36}\eta + \tfrac{1}{1620}\eta^2 - \tfrac{7}{6480}\eta^3 + \tfrac{5}{18144}\eta^4 - \tfrac{11}{382725}\eta^5 - \tfrac{101}{16329600}\eta^6 + \cdots,\\
\epsilon_2 &= -\tfrac{7}{405} - \tfrac{7}{2592}\eta + \tfrac{533}{204120}\eta^2 - \tfrac{1579}{2099520}\eta^3 + \tfrac{109}{1749600}\eta^4 + \tfrac{10217}{251942400}\eta^5 + \cdots,\\
\epsilon_3 &= +\tfrac{449}{102060} - \tfrac{63149}{20995200}\eta + \tfrac{29233}{36741600}\eta^2 + \tfrac{346793}{5290790400}\eta^3 + \cdots,
\end{aligned}$$

again with η replaced by η_0.

When we have obtained η from (3.10), we can compute λ by inverting the relation between η and λ in (2.4). For small values of η we can use a series expansion of λ in terms of η. Inverting

$$\tfrac{1}{2}\eta^2 = \tfrac{1}{2}(\lambda-1)^2 - \tfrac{1}{3}(\lambda-1)^3 + \tfrac{1}{4}(\lambda-1)^4 + \cdots,$$

Table 1
First five pairs $z_k^{\pm} = x_k \pm iy_k$ of zeros of erfc z.

k	x_k	$\pm\ iy_k$
1	−1.35481...	$\pm i$ 1.99146...
2	−2.17704...	$\pm i$ 2.69114...
3	−2.78438...	$\pm i$ 3.23533...
4	−3.28741...	$\pm i$ 3.69730...
5	−3.72594...	$\pm i$ 4.10610...

we obtain

$$\lambda = 1 + \eta + \tfrac{1}{3}\eta^2 + \tfrac{1}{36}\eta^3 - \tfrac{1}{270}\eta^4 + \tfrac{1}{4320}\eta^5 + \dots .$$

4. More information on the zeros and numerical examples

From Fettis et al. [1] we know that two infinite strings of zeros of erfc z occur in the neighbourhood of the diagonals $y = \pm x$ in the left half plane $x < 0$, $z = x + iy$. The first few zeros are given in table 1. Numerical values of the first 100 zeros of erfc z and asymptotic approximations of the zeros are also given by Fettis et al. A first order approximation reads

$$z_k^{\pm} \sim (-1 \pm i)\sqrt{(k - 1/8)\pi}. \tag{4.1}$$

When η is a zero of equation (3.9) and we use (3.10) as an asymptotic expansion of η for large positive values of a, we find that the λ-zeros of $Q(a, \lambda a)$ in terms of the parameter η occur in the neighbourhood of the diagonals $\mathcal{R}\eta = \pm\mathcal{I}\eta$, with $\mathcal{R}\eta < 0$. It is therefore of interest to know the original λ-contours of these diagonals under the mapping (2.3).

To study the mapping and the pre-images of the diagonals, let us write $\eta = \alpha + i\beta$ and $\lambda = \rho\exp(i\phi)$. Then the relation between η and λ given in (2.3) can be written in terms of the real equations

$$\begin{aligned} \tfrac{1}{2}(\alpha^2 - \beta^2) &= \rho\cos\phi - 1 - \ln\rho, \\ \alpha\beta &= \rho\sin\phi - \phi. \end{aligned} \tag{4.2}$$

On the diagonals $\alpha = \pm\beta$ the first equation becomes $\rho\cos\phi = 1 + \ln\rho$. In terms of Cartesian coordinates ($\lambda = x + iy$) we have

$$x^2 + y^2 = e^{2(x-1)}. \tag{4.3}$$

This equation defines an almond-shaped closed curve between $x = 1$ and $x = -0.278\dots$, the solution of the equation $-x = \exp(x - 1)$; see figure 1. In [6] this curve is called the *Szegö curve* and defined as

$$D_\infty = \{\lambda \in \mathbb{C}\colon |\lambda e^{1-\lambda}| = 1 \quad \text{and} \quad |\lambda| \leqslant 1\}.$$

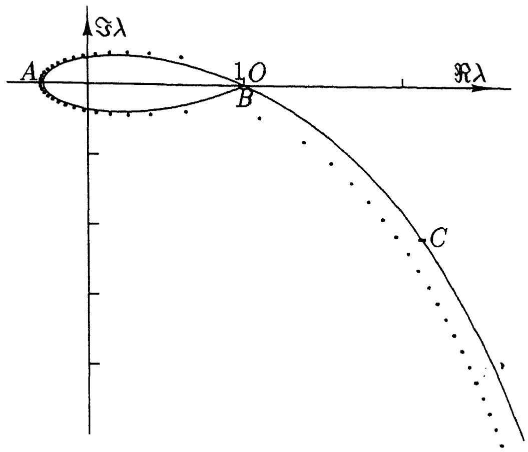

Figure 1. λ-zeros of $\Gamma(a, \lambda a)$ with $a = 30.1$, along curves defined by $x^2 + y^2 = \exp[2(x-1)]$; shown are the zeros with phases in the interval $[0, 2\pi]$. There is a conjugate set of zeros with phases in the interval $[-2\pi, 0]$.

As discussed in detail in [6], when $a = n$ (integer) the n λ-zeros of $\Gamma(n, \lambda n)$ (a polynomial in λ, see the second line in (1.3)) approach this curve when $n \to \infty$.

The equation in (4.3) also defines solutions for $x > 1$; there are two branches starting in $x = 1$, $y = 0$ and extending to infinity along the curves $y = \pm\exp(x-1)$. These branches have no meaning in connection with the zeros of $\Gamma(a, \lambda a)$ when a is an integer. However, when a is not an integer $\Gamma(a, \lambda a)$ has an infinity of λ-zeros of which about $[a]$ zeros are located near D_∞ with phases in the interval $(-\pi, \pi)$. An infinite number of zeros is located near the branches of equation (4.3) with $x > 1$, but the phases of the zeros are in the intervals $(-2\pi, -\pi)$ and $(\pi, 2\pi)$.

To see this, we give more details on the mapping given in (4.2). Finite singular points of the mapping $\lambda \mapsto \eta(\lambda)$ can be found by considering the derivative $d\eta/d\lambda = (\lambda - 1)/(\eta\lambda)$. The point $\lambda = 1$ is a regular point, but $\lambda = \exp(2\pi i n)$ $(n = \pm 1, \pm 2, \ldots)$ are points where the derivative $d\eta/d\lambda$ vanishes. They correspond to the η-values satisfying $\eta_n^2/2 = -2\pi i n$, giving the singular points $\eta^\pm = 2\sqrt{\pi}\exp(\pm 3\pi i/4)$ for $n = 1$. The points $\eta^\pm$ are located at the diagonals in the left half plane, and they correspond to $\lambda = \exp(\pm 2\pi i)$. It can be proved (details will not be given here) that the λ-sector $|\arg\lambda| < 2\pi$, $(\lambda \neq 0)$ is mapped conformally and one-to-one onto the η-plane cut along two branch lines, which are parts of the hyperbolas $\alpha\beta = \pm 2\pi$ with $\alpha \leqslant -\sqrt{2\pi}$. The boundaries of this sector, that is, the half lines $\arg\lambda = \pm 2\pi i$ are mapped onto these branch cuts. For more details we refer to [3].

To understand the role of the diagonals in the left half η-plane in connection with the location of the zeros of the incomplete gamma function $\Gamma(a, \lambda a)$ in the λ-plane, we concentrate on a few points O, A, B, C on the diagonal $\alpha = -\beta$, $\beta \geqslant 0$ (because of the symmetry it is sufficient to consider $\beta \geqslant 0$). Let O denote the origin, A the point where $\alpha\beta = -\pi$, B the point where $\alpha\beta = -2\pi$, and C a point where $\alpha\beta < -2\pi$. Then the segment $[O, A]$ corresponds to the upper part of the almond-shaped curve in figure 1, $[A, B]$ corresponds to the lower part of this curve (the phase of λ is now in the interval $[\pi, 2\pi]$), and $[B, C]$ with part of the branch extending to the right of $\lambda = 1$, with $\mathcal{I}\lambda < 0$ (and with the phase of λ still in $[\pi, 2\pi]$).

This description of the mapping with respect to the diagonals in the left η-plane makes clear how the approximations of the zeros of $\Gamma(a, \lambda a)$ in the η-plane are mapped onto the λ-plane. The η-zeros in the neighbourhood of the segment $[O, A]$ are mapped onto the upper part of the almond-shaped curve in the λ-plane. From the estimates given for the zeros of the error function, see (4.1), we see that there are about $a/2$ zeros along this curve. Another $a/2$ are located along the lower part of the almond (with phases belonging to $[\pi, 2\pi]$); they correspond to η-values in the neighbourhood of the segment $[A, B]$. When $a = n$ (integer) there are exactly n zeros along the complete almond. In that case it is not needed to prescribe the phases of the zeros on the lower part of the almond, since then $\Gamma(a, \lambda a)$ is not multi-valued.

The zeros corresponding to η-zeros along the diagonal $\alpha = \beta$ in the third quadrant of the η-plane yield a conjugate set of zeros in the λ-plane. The diagonals in the right half plane $\alpha \geqslant 0$ correspond to the branches in the λ-plane that start in the point $\lambda = 1$ and extend to $\pm i\infty$ along the curves $y = \pm\exp(x - 1)$, with phases of λ belonging to $[0, \pm\pi/2]$. Along these contours in the λ-plane we find zeros of $\gamma(a, \lambda a)$. This follows from the minus sign in the error function representation of the function $P(a, z)$ in (2.1).

In a separate publication (a first orientation on the computation of incomplete gamma functions for complex parameters is given in [5]) we will verify numerically the asymptotic expansions of the zeros of $\Gamma(a, \lambda a)$ and $\gamma(a, \lambda a)$ derived in this paper.

In figure 1 we show the first 50 zeros of $\Gamma(a, \lambda a)$ with $a = 30.1$ with phases in the interval $(0, 2\pi)$. In the upper half plane the phases of the zeros are in the interval $(0, \pi)$, in the lower half plane they are in the interval $(\pi, 2\pi)$. A similar picture can be given for zeros with phase in $(-2\pi, 0)$. When a is an integer the function $\Gamma(a, \lambda a)$ is single-valued, and then we can consider phases just in the interval $(-\pi, \pi)$. In that case the zeros along the branches extending to infinity do not occur.

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